Temporal and spatial variations of bed stresses

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Digressions:

- **Tsunami research is an interdisciplinary research:** **fluid mechanics; seismology; geophysics; geology; structural/geotech engineering; social science**

- **State of tsunami modeling:**

  Source region: primitive **(but good enough for early warning)**

  Propagation: satisfactory

  Coastal effects: many open questions **(e.g. tsunami wave form; dissipative mechanism; sediment transport; debris flows; wave-structure interaction etc.)**

Background and Motivations:

- In recent years, various phase-resolving, depth-integrated wave models, including the shallow-water equations and Boussinesq equations, have been successfully developed to simulate wave hydrodynamics in the nearshore region.

- These models are being used as “wave drivers” for calculating the bed stress, the sediment (suspended and bedload) transport and the corresponding morphological changes.

- These models are based on the potential flow assumption in which seabed is usually assumed to be smooth and impermeable. Various seabed characteristics have not been properly considered.

Bed stress

Traditionally, the bed stress is modeled as a quadratic function of the bottom velocity, i.e.,

$$\tau_b = \rho C_f \left| u_b \right| u_b , \quad C_f = f(R_e)$$

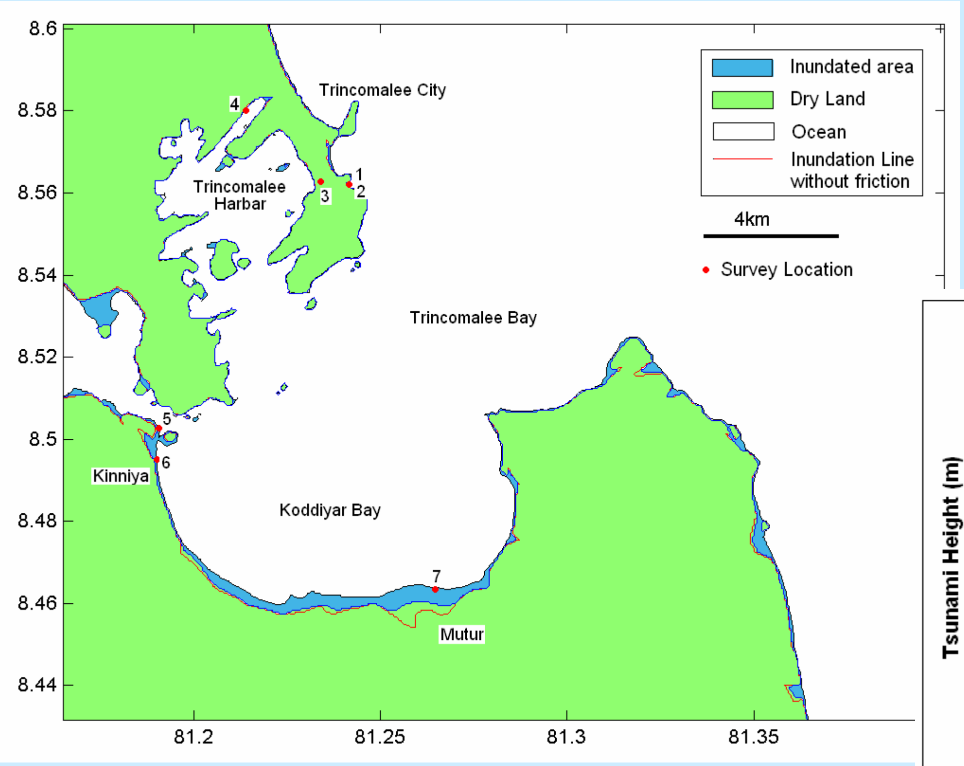
If the effects of bed stress on the wave propagation are desired, the bottom stress is added in the horizontal momentum equations.

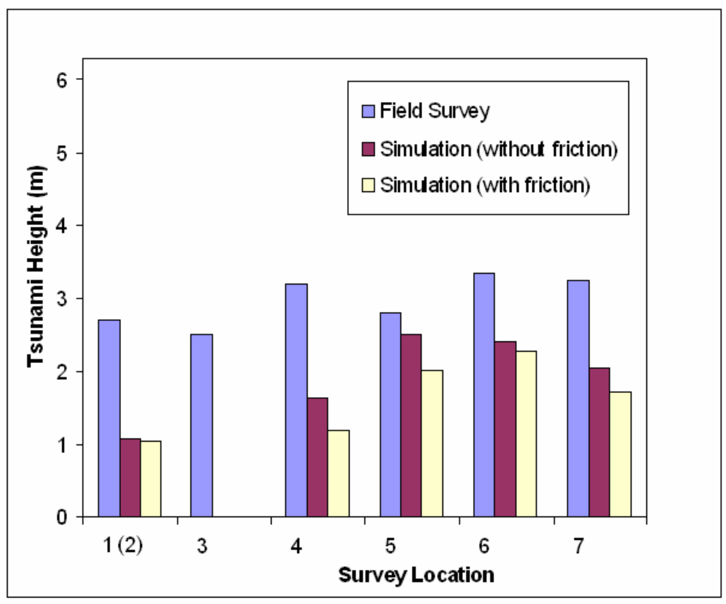
Conventional non-linear shallow-water equations with bottom stress considered

$$\frac{\partial H}{\partial t} + \nabla \cdot \left(H \bar{\boldsymbol{u}} \right) = 0$$

$$\frac{\partial \bar{\boldsymbol{u}}}{\partial t} + \bar{\boldsymbol{u}} \cdot \nabla \bar{\boldsymbol{u}} + g \nabla \zeta + \frac{C_f \left| \bar{\boldsymbol{u}} \right| \bar{\boldsymbol{u}}}{\rho h} = 0$$

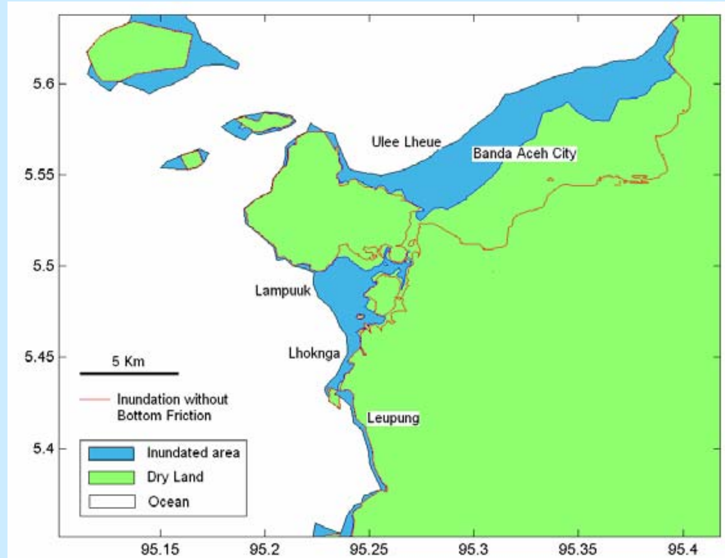
Tsunami inundation in Trincomalee (red line shows the inundation line)



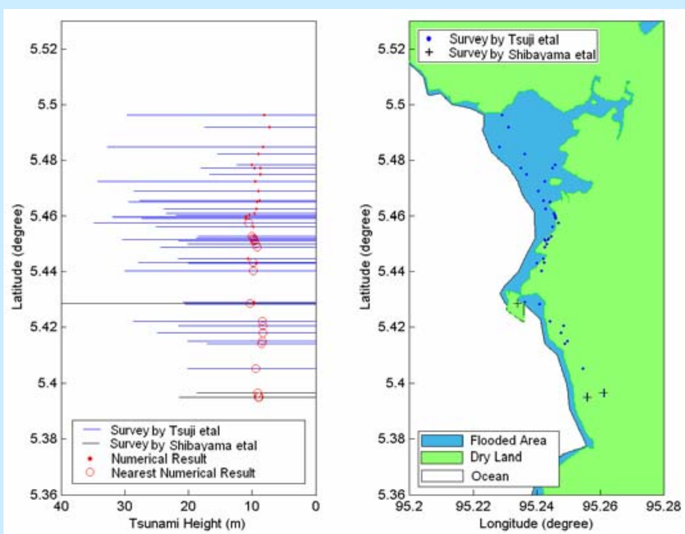


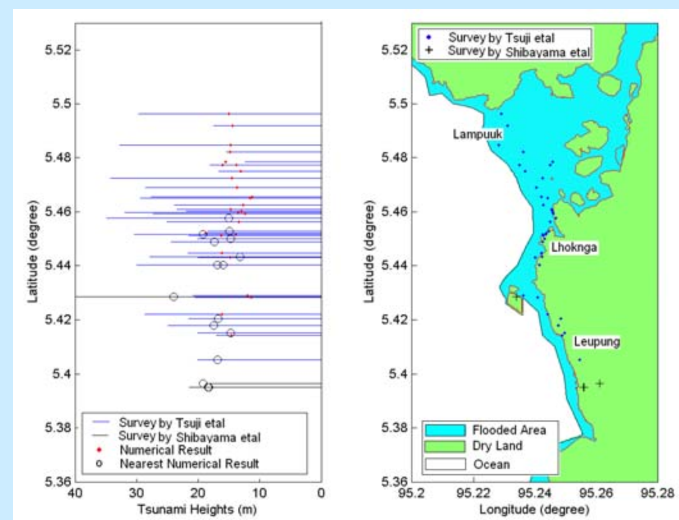
Comparisons between survey data and numerical results

Tsunami Runup and Inundation in Banda Aceh



Calculated inundation area





Comparisons between survey data and numerical results

Sediment transport

Shield parameter

$$\theta = \frac{\tau_b}{(\rho_s - \rho) g d_s}, \quad \tau_b = \sqrt{\tau_x^2 + \tau_y^2}$$

$\theta > 0.06$: incipience of grain movement

Sediment transport rate (bed load)

$$\vec{q} = 8\sqrt{\frac{\rho_s - \rho}{\rho} g d_s} \Phi \frac{\vec{u}}{|\vec{u}|}; \quad \Phi = (\theta - \theta_c)\sqrt{|\theta|}$$

Morph-dynamic equation

$$\frac{\partial h}{\partial t} + \frac{1}{1-\lambda} \nabla \cdot \vec{q} = 0$$

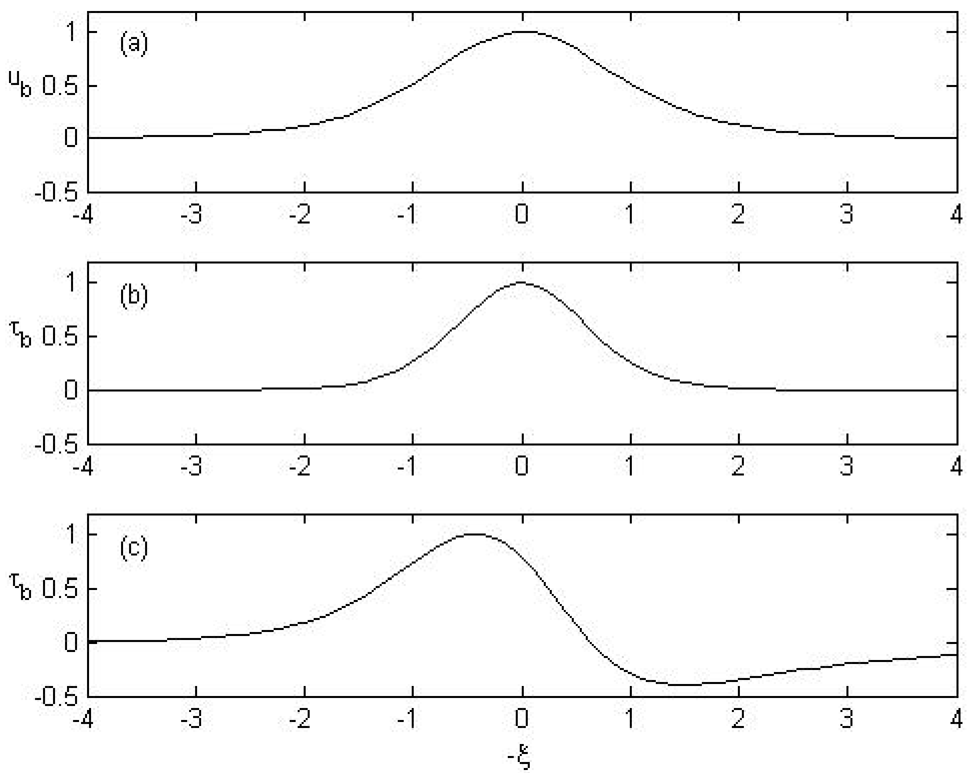
What are the issues facing the conventional approach?

- It is known that the bed stress and the free-stream velocity is 45 degree out of phase for an oscillatory laminar boundary layer
- For turbulent boundary layer, the phase lag is about 10 degree
- The effects of velocity deficit in the boundary layer are not considered in the depth-integrated continuity equation
- Since the sediment transport (bedload) is directly related to the bottom stress (frictional velocity), the phase of the bottom stress under a transient wave train has significant consequences on morphological dynamics

SOLITARY WAVE

$$\tau_b \approx |u|u$$

$$\tau_b = \frac{1}{\sqrt{\pi}} \int_0^t \frac{\partial \bar{\boldsymbol{u}} / \partial \tau}{\sqrt{t-\tau}} d\tau$$



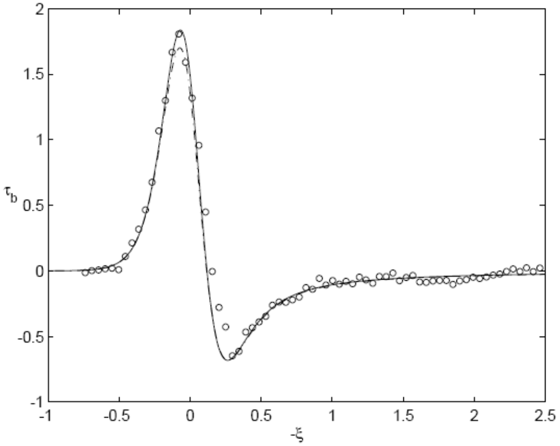


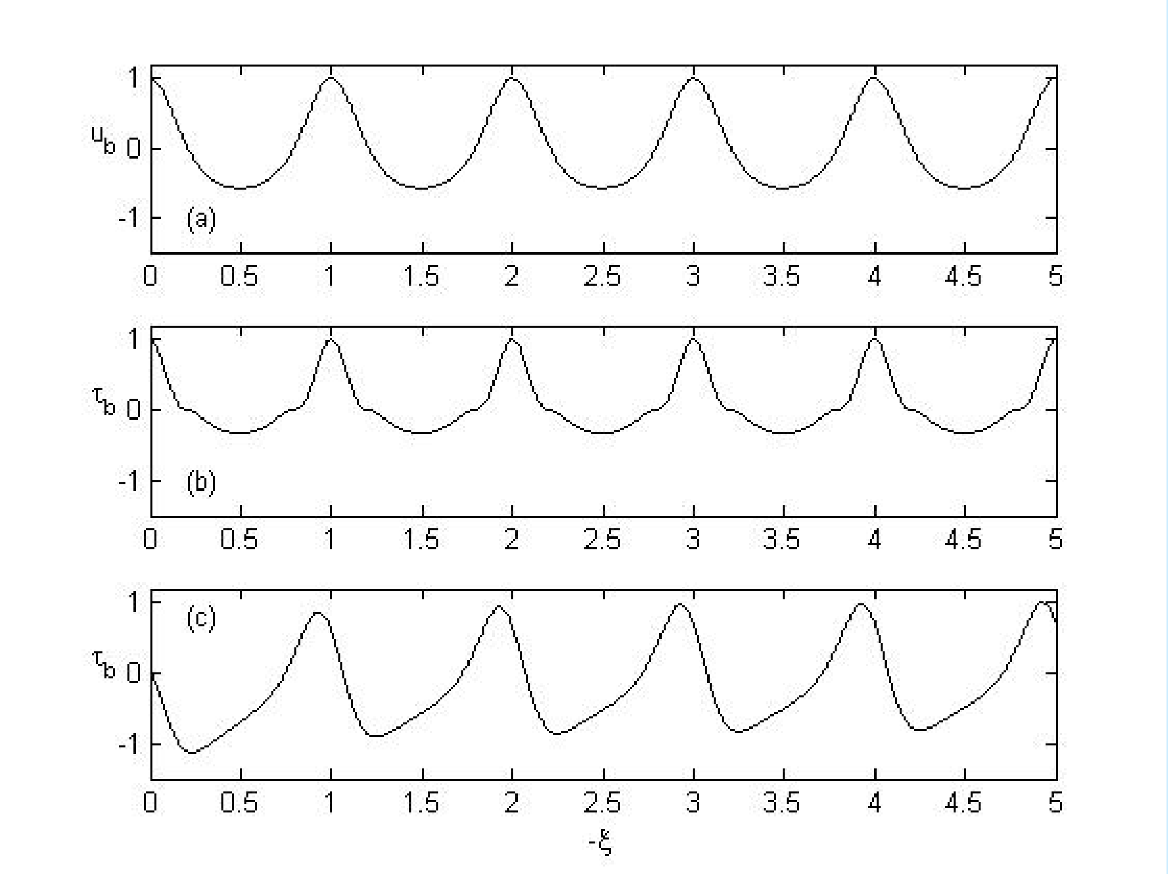
FIGURE 11. Time history of the bottom shear stress for $\epsilon = 0.2$. —— for the nonlinear solutions; —·— for the linear solutions (2.21); ○○○ for the experimental data.

(Liu *et al*. 2006 *JFM*)

Cnoidal Waves

$$\tau_b \approx |u|u$$

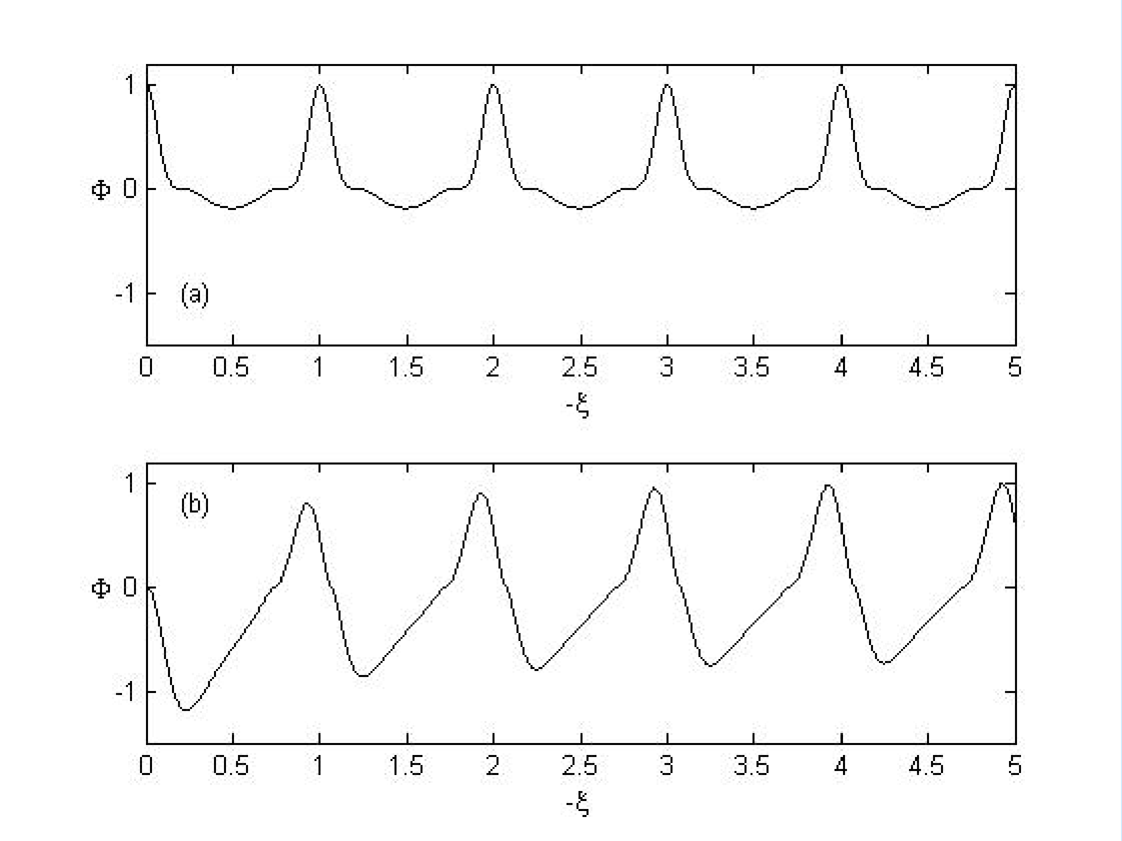
$$\tau_b = \frac{1}{\sqrt{\pi}} \int_0^t \frac{\partial \bar{\mathbf{u}} / \partial \tau}{\sqrt{t-\tau}} d\tau$$



Sediment transport rate: $\vec{q} = 8\sqrt{\frac{\rho_s - \rho}{\rho} g d_s} \Phi \frac{\vec{u}}{|\vec{u}|}; \quad \Phi = (\theta - \theta_c)\sqrt{|\theta|}$

$$\tau_b \approx |u|u$$

$$\tau_b = \frac{1}{\sqrt{\pi}} \int_0^t \frac{\partial \overline{\mathbf{u}} / \partial \tau}{\sqrt{t-\tau}} d\tau$$

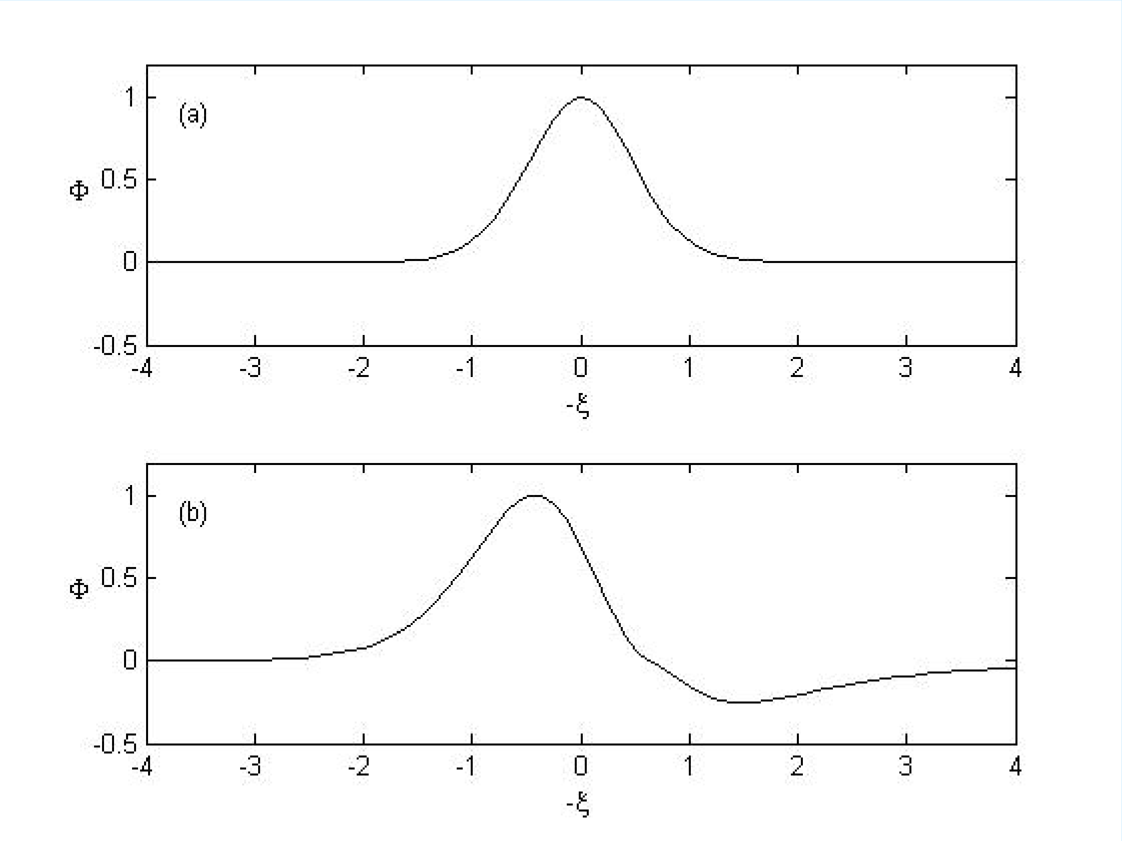


The net transport rates over a period have different signs

Sediment transport rate: Solitary wave

$$\tau_b \approx |u|u$$

$$\tau_b = \frac{1}{\sqrt{\pi}} \int_0^t \frac{\partial \bar{\mathbf{u}} / \partial \tau}{\sqrt{t-\tau}} d\tau$$



Boussinesq equations with boundary layer effects considered:

$$\frac{1}{\varepsilon}\frac{\partial H}{\partial t}+\nabla\cdot\left(H\overline{\boldsymbol{u}}\right)-\frac{\alpha}{\mu^2}\frac{1}{\sqrt{\pi}}\int_0^t\frac{\nabla\cdot\overline{\boldsymbol{u}}}{\sqrt{(t-\tau)}}d\tau=0\left(\mu^4\right)$$

$$\frac{\partial\overline{\boldsymbol{u}}}{\partial t}+\varepsilon\overline{\boldsymbol{u}}\cdot\nabla\overline{\boldsymbol{u}}+\frac{1}{\varepsilon}\nabla H+\mu^2\left[\frac{1}{6}\nabla^2\left(\frac{\partial\overline{\boldsymbol{u}}}{\partial t}\right)-\frac{1}{2}\nabla\nabla\cdot\left(\frac{\partial\overline{\boldsymbol{u}}}{\partial t}\right)\right]=0\left(\mu^4\right)\quad .$$

Dimensionless Parameters:

$$\varepsilon=a_0'/h',\quad \mu=h'/l_o',\quad \alpha^2=\frac{\nu}{l_o'\sqrt{gh'}}$$

Bed stress

$$\tau_b=\left.\frac{\partial u_0^r}{\partial\eta}\right|_{\eta=0}=\frac{1}{\sqrt{\pi}}\int_0^t\frac{\partial\overline{\boldsymbol{u}}/\partial\tau}{\sqrt{t-\tau}}d\tau,\qquad \eta=\frac{z+1}{\alpha}$$

One-dimensional case:

$$\frac{\partial \zeta}{\partial t}+\frac{\partial}{\partial x}\left[\left(1+\varepsilon\zeta\right)\overline{u}\right]-\frac{\alpha}{\mu^2}\frac{1}{\sqrt{\pi}}\int_0^t \frac{\partial \overline{u}}{\partial x}\frac{1}{\sqrt{t-\tau}}d\tau=0,$$

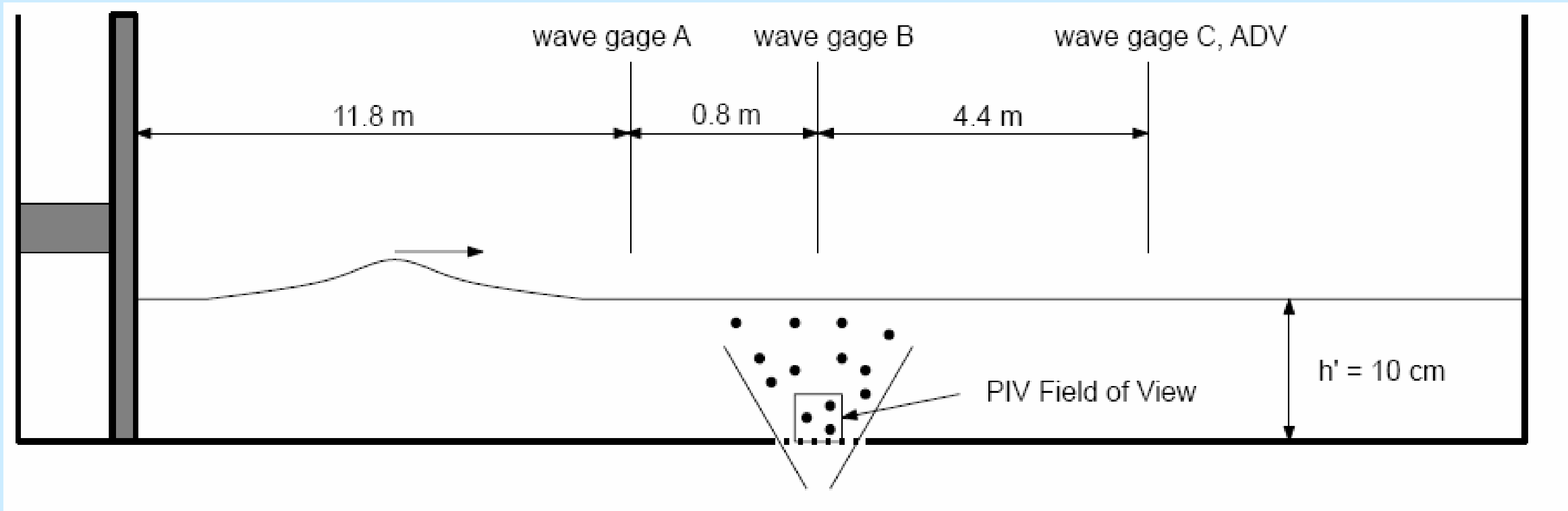
$$\frac{\partial \overline{u}}{\partial t}+\varepsilon\overline{u}\frac{\partial \overline{u}}{\partial x}+\frac{\partial \zeta}{\partial x}-\frac{\mu^2}{3}\frac{\partial^3 \overline{u}}{\partial x^2 \partial t}=0.$$

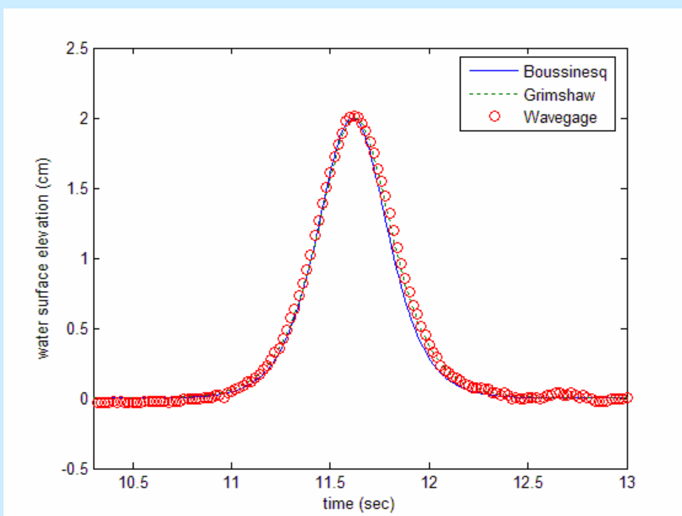
In the moving coordinate: $\sigma = x - t, \quad \overline{\xi} = \varepsilon t$

KdV equations

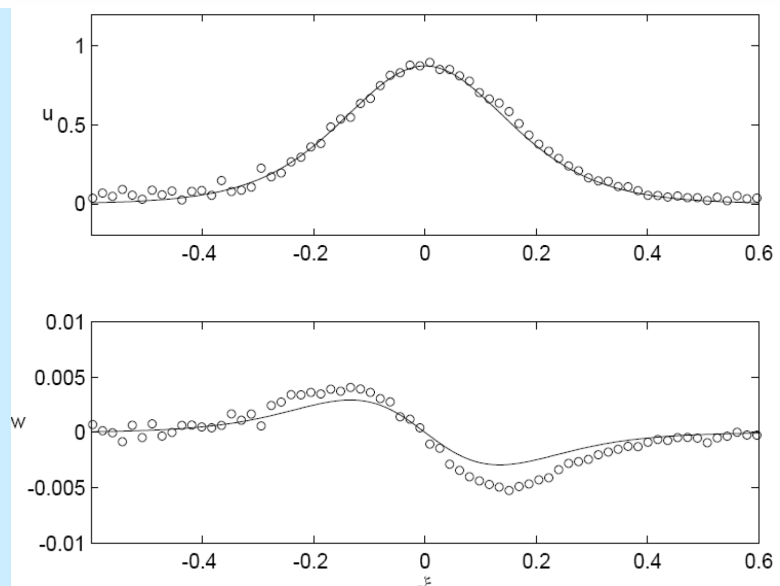
$$\frac{\partial \zeta}{\partial \overline{\xi}}+\frac{3}{2}\zeta\frac{\partial \zeta}{\partial \sigma}+\frac{1}{6}\frac{\mu^2}{\varepsilon}\frac{\partial^3 \zeta}{\partial \sigma^3}=\frac{\alpha}{\mu^2\varepsilon}\frac{1}{2\sqrt{\pi}}\int_0^t \frac{\partial \zeta}{\partial x}\frac{1}{\sqrt{t-\tau}}d\tau$$

Validations of the boundary layer formulation with PIV data





Free surface displacement $\varepsilon = 0.2$



Theoretical and experimental water particle velocities in the irrotational region for $\epsilon = 0.2$. —— for Grimshaw's solution; ○○○ for experimental data.

Validations of the boundary layer formulation with PIV data

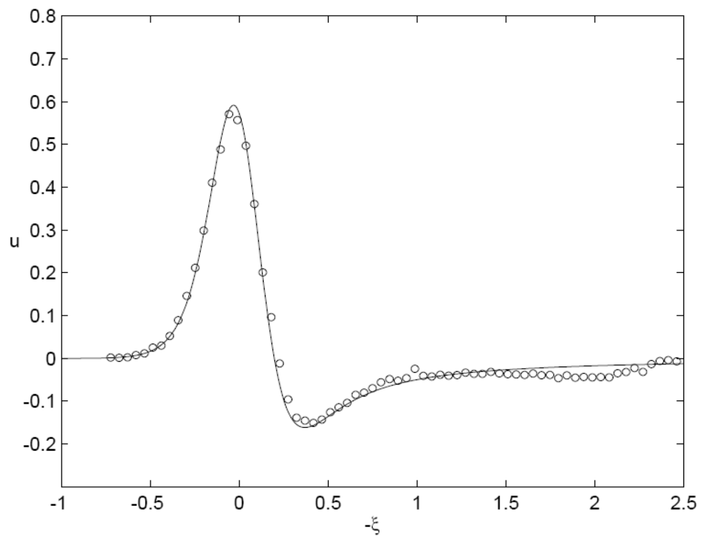
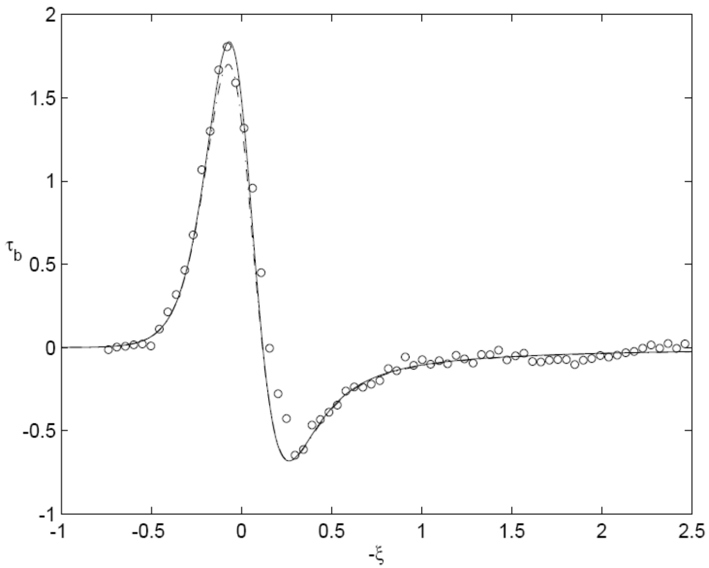
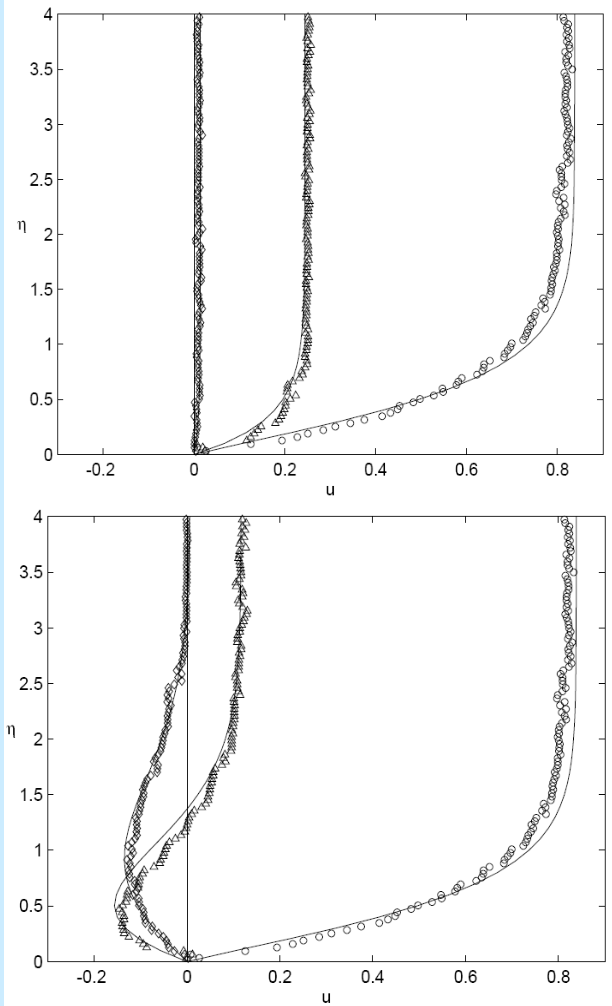


FIGURE 10. Time history of the horizontal velocity at $\eta = 0.5$ for $\epsilon = 0.2$. —— for the nonlinear solutions; ○○○ for the experimental data.



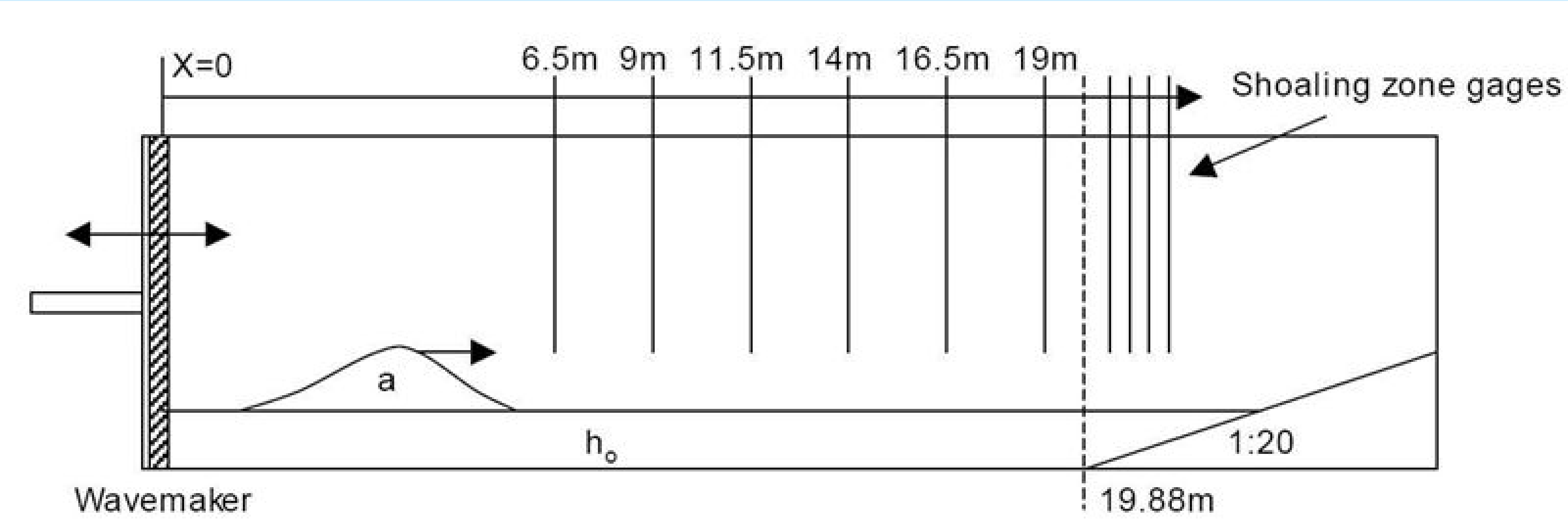
Bottom stress



Vertical profiles of horizontal velocity inside the bottom boundary layer

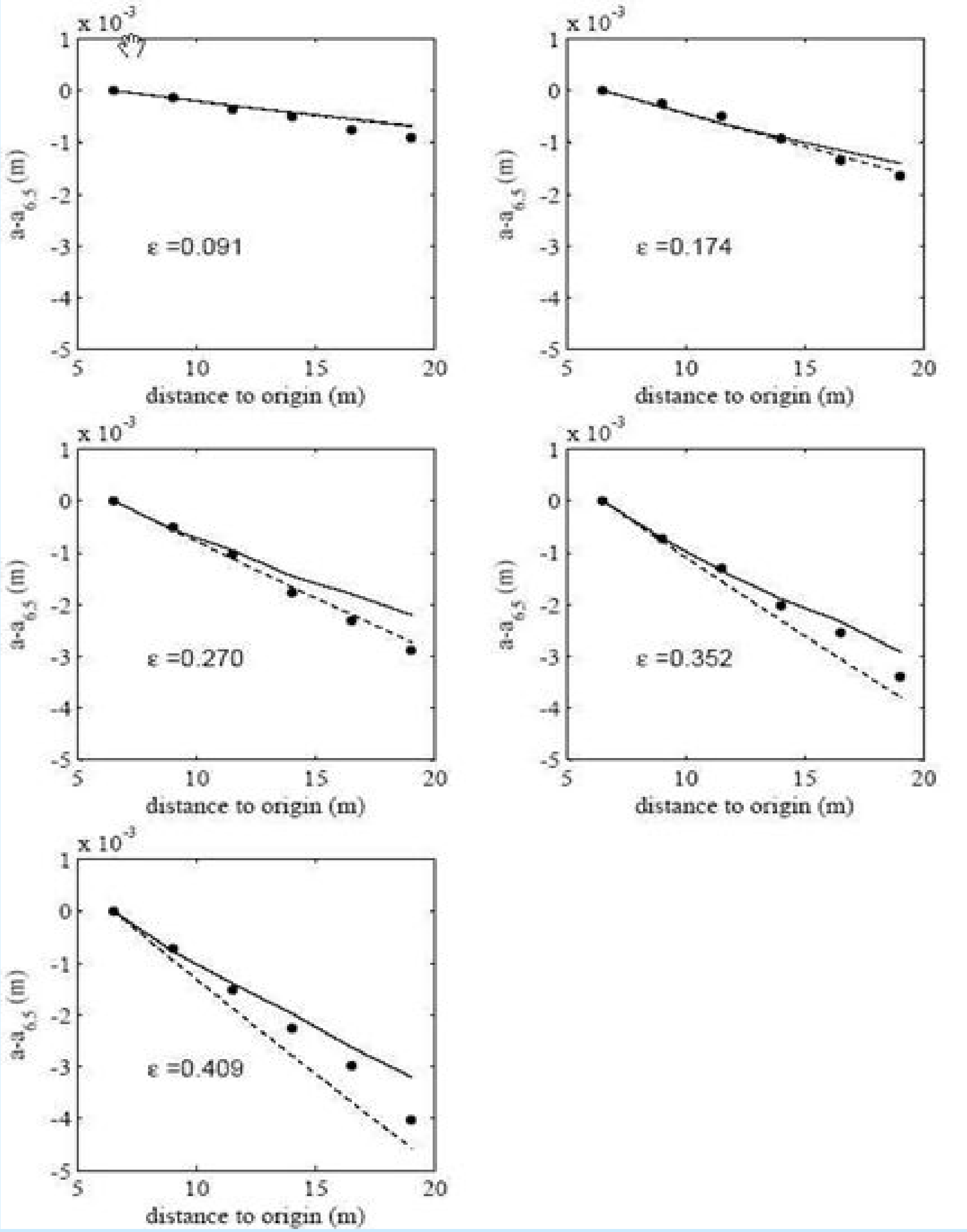
(Liu *et al. JFM* 2006)

Damping of solitary wave amplitude along a wave tank



(Liu *et al.* 2005 *Coastal Engineering*)

Damping of solitary wave amplitude along the wave tank



$$\zeta = a(\xi)\operatorname{sech}^2\left[\frac{\sqrt{3a}}{2}\rho\right]$$

$$1 - a^{-1/4} = -0.0836\xi$$

Turbulent Boundary Layer

Eddy viscosity model

$$\tau' = \rho \nu_e \frac{\partial \boldsymbol{u}'^r}{\partial z'}, \qquad \nu_e = b' u_* \left(z' + h' \right)^p, \qquad 0 \le p < 1$$

Bottom stress

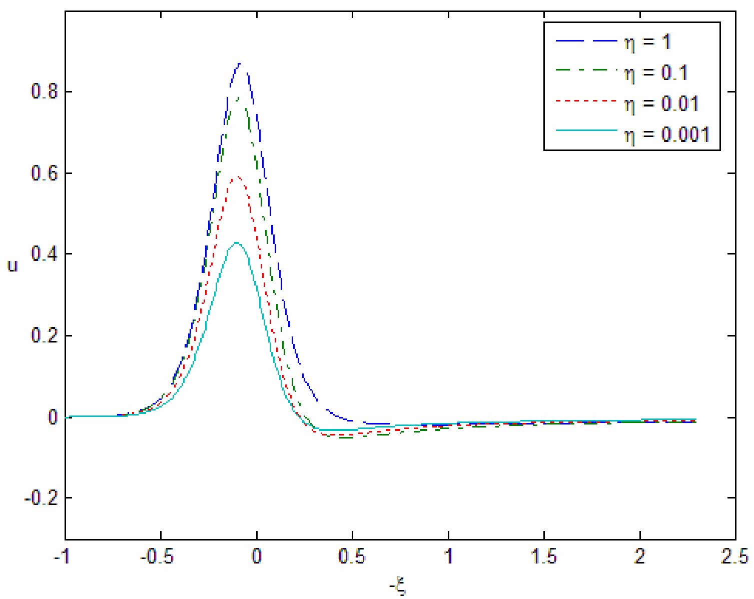
$$\tau_b = -\frac{q(1-q)^{2q-1}}{\Gamma(q+1)} \int_0^t \frac{\partial \overline{\boldsymbol{u}} / \partial \tau}{\left(t - \tau \right)^{1+q}} d\tau, \qquad q = (1-p)/(2-p)$$

Special cases: (1) a constant viscosity with *p* = 0, and (2) Prandtl's one-seventh power law approximation for a turbulent boundary layer velocity with *p* = 6/7.

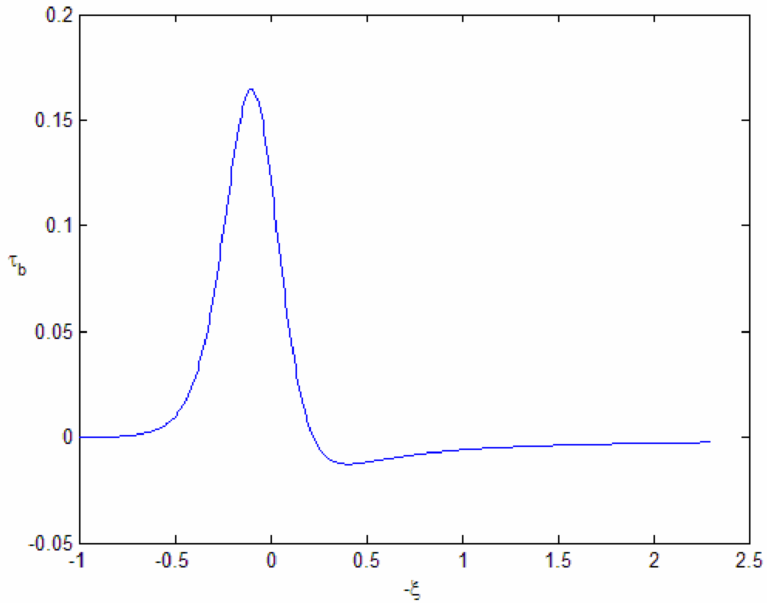
Boussinesq equations with Turbulent Boundary Layer:

$$\frac{1}{\varepsilon}\frac{\partial H}{\partial t}+\nabla\cdot\left(H\bar{\boldsymbol{u}}\right)-\frac{\alpha}{\mu}\frac{q(1-q)^{2q-1}}{\Gamma\left(q+1\right)}\int_0^t\frac{\nabla\cdot\bar{\boldsymbol{u}}}{\left(t-\tau\right)^q}d\tau=0\left(\mu^4\right)$$

$$\frac{\partial\bar{\boldsymbol{u}}}{\partial t}+\varepsilon\bar{\boldsymbol{u}}\cdot\nabla\bar{\boldsymbol{u}}+\frac{1}{\varepsilon}\nabla H+\mu^2\left[\frac{1}{6}\nabla^2\left(\frac{\partial\bar{\boldsymbol{u}}}{\partial t}\right)-\frac{1}{2}\nabla\nabla\cdot\left(\frac{\partial\bar{\boldsymbol{u}}}{\partial t}\right)\right]=0\left(\mu^4\right)\quad .$$



Time histories of horizontal velocity at different elevations for Prandtl's turbulence boundary layer



Time history of bottom stress inside Prandtl's boundary layer

Boussinesq equations with a viscous fluid mud bed

$$\frac{1}{\varepsilon}\frac{\partial H}{\partial t}+\nabla\cdot\left[(H+\gamma d)\mathrm{u}_I\right]-\frac{\mu^2}{6}\nabla^2\nabla\cdot\mathrm{u}_I-\gamma\frac{\alpha}{\mu}\int_0^t\nabla\cdot\frac{\partial\mathrm{u}_I}{\partial\tau}\sqrt{4(t-\tau)}I(t-\tau)d\tau=\mathrm{O}(\mu^4)$$

$$\frac{\partial\mathrm{u}_I}{\partial t}+\varepsilon\mathrm{u}_I\cdot\nabla\mathrm{u}_I+\frac{1}{\varepsilon}\nabla H-\frac{\mu^2}{2}\frac{\partial}{\partial t}\nabla\nabla\cdot\mathrm{u}_I=\mathrm{O}(\mu^4)$$

where $\gamma=\rho/\rho_m$

$$\begin{aligned}I(t-\tau)=&\frac{1}{\sqrt{\pi}}\left[1-\exp\left(\frac{-\bar{d}^2}{4(t-\tau)}\right)\right]+\frac{\bar{d}}{\sqrt{4(t-\tau)}}erfc\left(\frac{\bar{d}}{\sqrt{4(t-\tau)}}\right)\\&-\sum_{n=1}^{\infty}\left\{\frac{1}{\sqrt{\pi}}\left[-4\exp\left(-\frac{(2n\bar{d})^2}{4(t-\tau)}\right)+\sum_{m=-1}^{1}(2+(-1)^m)\exp\left(-\frac{((2n-m)\bar{d})^2}{4(t-\tau)}\right)\right]\right.\\&\left.+\frac{\bar{d}}{\sqrt{4(t-\tau)}}\left[2nerfc\left(\frac{2n\bar{d}}{\sqrt{4(t-\tau)}}\right)+\sum_{m=-1}^{1}(-1)^m(2n-m)erfc\left(\frac{(2n-m)\bar{d}}{\sqrt{4(t-\tau)}}\right)\right]\right\}\end{aligned}$$

Horizontal velocity component in the mud layer under a solitary wave

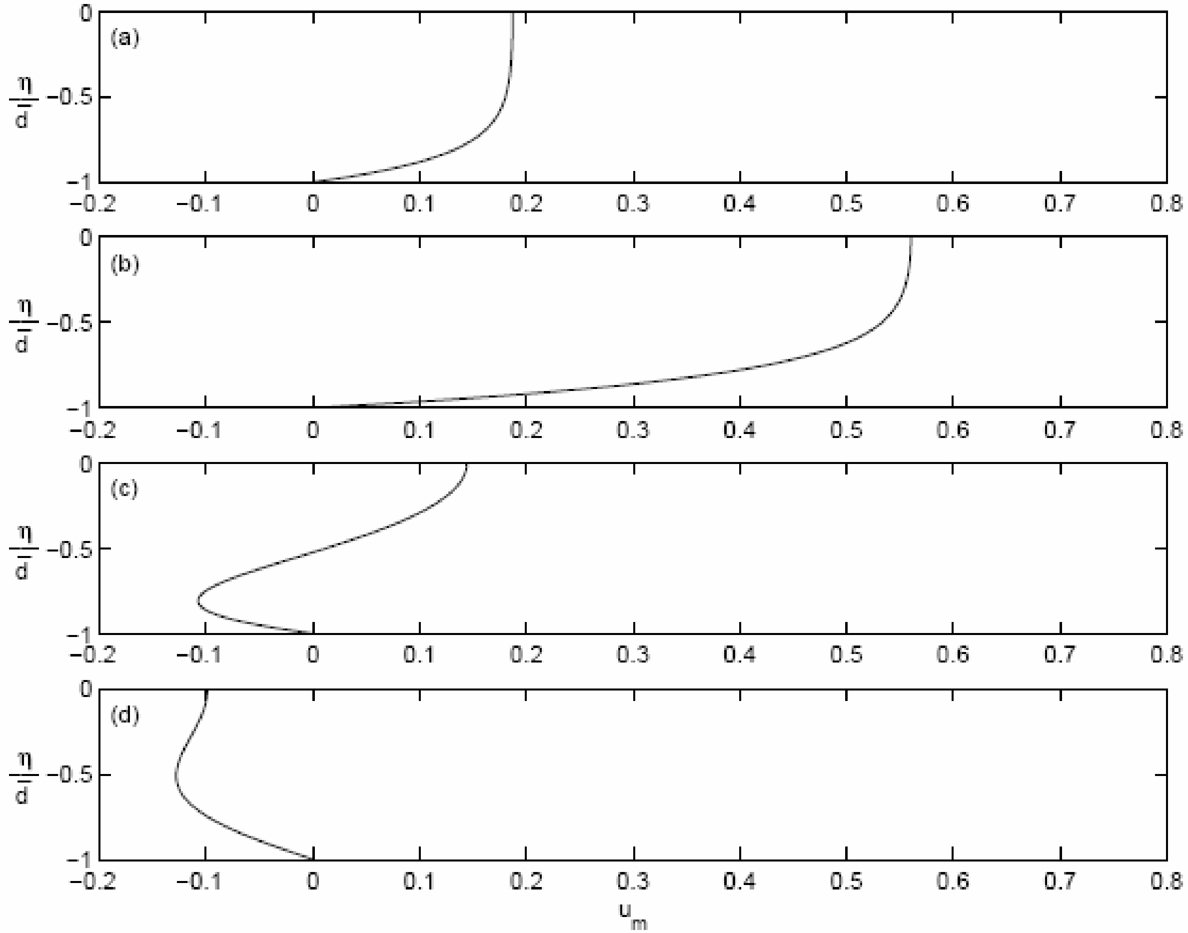


FIGURE 1. Horizontal velocity inside the mud bed at different phases under a solitary wave. (a) $u_b = 0.25$ during the acceleration phase; (b) $u_b = 0.75$ during the acceleration phase;(c) $u_b = 0.25$ during the deceleration phase; (d) $u_b = 0.01$ during the deceleration phase. ($\gamma = 0.75, \epsilon = \mu^2 = 0.1, \alpha = 0.01, \overline{d} = 5, x_0 = -50$)

Damping of solitary wave over a viscous fluid mud bed

$$\zeta = a(\xi)\,\text{sech}^2\left[\frac{\sqrt{3a}}{2}\rho\right]$$

$$\frac{da}{d\xi} = \frac{\sqrt{3}}{2}\gamma a^{3/2}\int_{-\infty}^{\infty}\int_{0}^{\infty}\text{sech}^2 R\,\text{sech}^4(R+S)\left[2-\cosh 2(R+S)\right]I\left(\frac{2S}{\sqrt{3a}}\right)dS\,dR$$

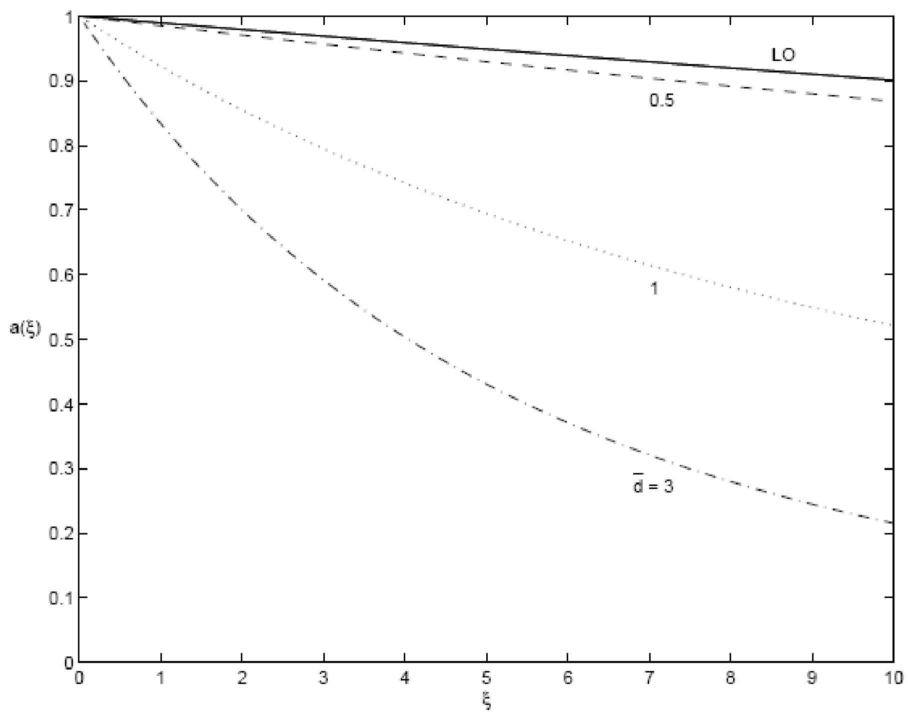


FIGURE 3. Wave amplitude, a, as a function of time, ξ. The solid line denotes the viscous damping caused by the bottom boundary layer with consideration of only water viscosity (Liu & Orfila 2004). Other curves represent the damping rates induced by the mud bed with different thickness. $\nu_m/\nu_w = 10^3$ has been used in the solutions.

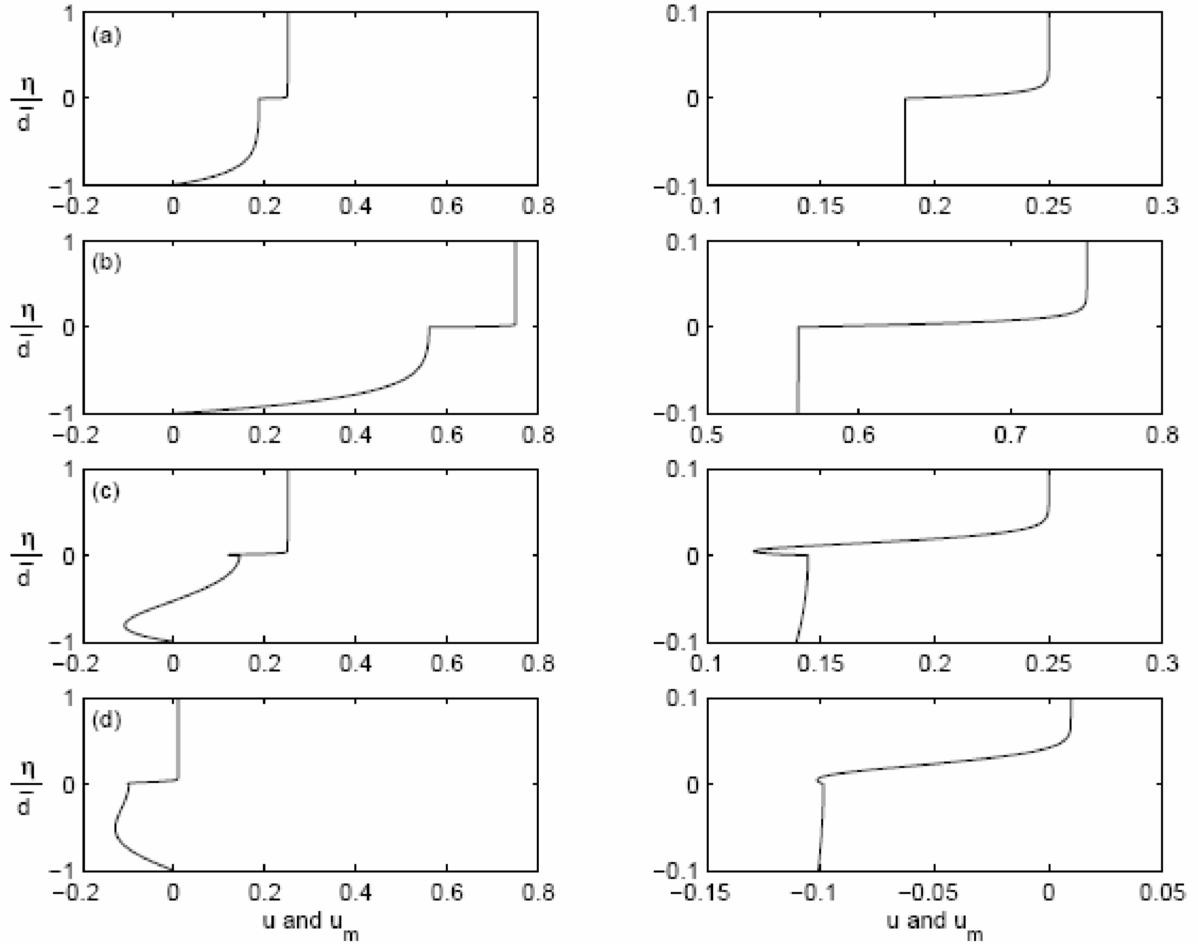


FIGURE 4. Horizontal velocity components at different phases with consideration of the interfacial boundary layer. The left column shows the velocity profiles in the entire mud bed and a portion of water depth above, while the right column presents the velocity profiles in the vicinity of interface between water and mud bed. The phase of each case is same as that shown in figure 1. Panel (a) and (b) denote the acceleration phases. Panel (c) and (d) represent the deceleration phases. $\nu_m/\nu_w = 10^3$ is used and other parameters are the same as those used in figure 1.

New Sheet Flow Measurements

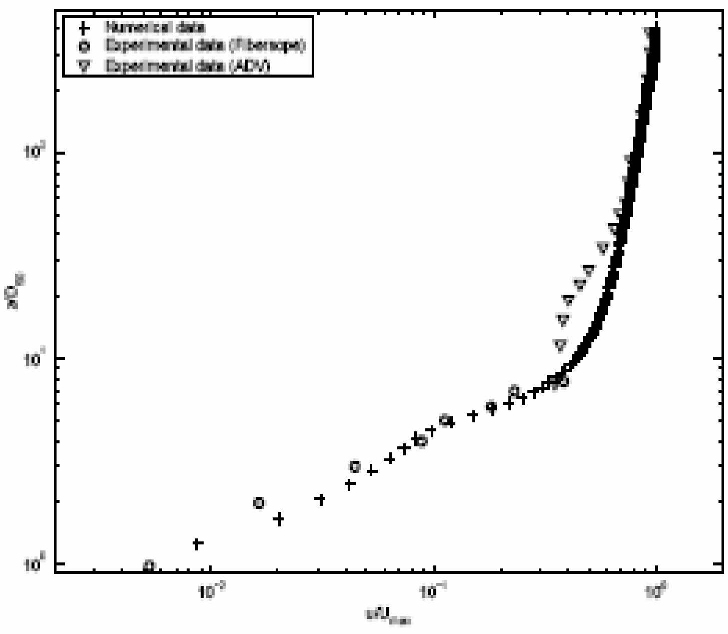


Figure 9: Model-data comparison of the horizontal sediment velocity for the whole water column.

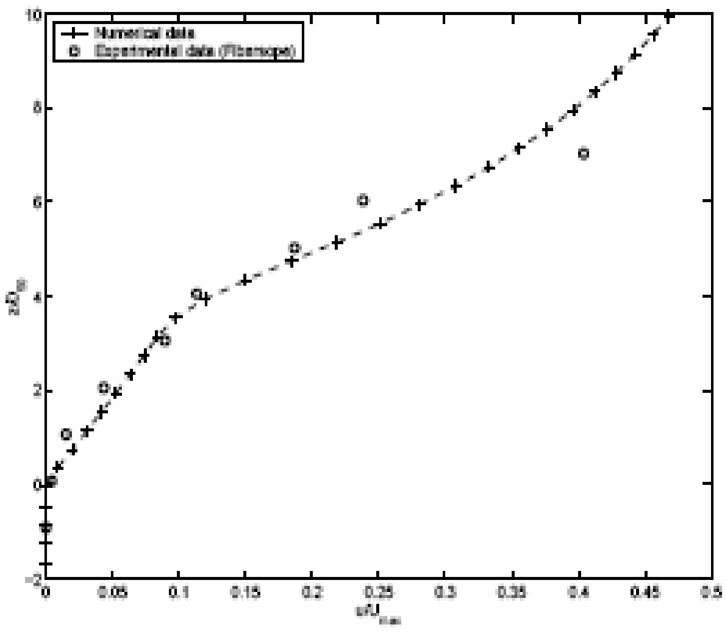


Figure 10: Model-data comparison of the horizontal sediment velocity in the near bed region.

Work in progress and future plans

- **Develop a methodology to include seafloor dynamics in the Boussinesq-type wave models**

  Options: 1. Analytical solutions (i.e., Liu & Orfila 2004);

  2. Multi-layer approach (i.e. Lynett & Liu 2004, Proc. Roy. Soc. London, A.)
- **Investigate effects of different seafloor dynamics, such as the poro-elastic materials, on long waves.**
- **Develop simple laboratory experiments to validate the theories and solutions.**

Thank you

Questions?

Comments?